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ABSTRACT LERAY–SCHAUDER TYPE ALTERNATIVES AND EXTENSIONS

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Abstract

We present a Leray–Schauder type alternative for a general class of maps. This enables us to obtain some Birkhoff–Kellogg type results and a Furi–Pera result.

1. Introduction.

In this paper coincidence theory of Leray–Schauder type for a general class of maps is presented and our theory extends and generalizes well known results in the literature (see [1, 3, 7, 8, 10] and the references therein). The argument presented is elementary and is based on an Urysohn type lemma. In addition we present an abstract Furi–Pera type fixed point theorem which extends and generalizes results in the literature (see [1, 6] and the references therein). Also our Leray–Schauder coincidence theory is used to establish some Birkhoff–Kellogg type theorems for a general class of maps.

2. Main results.

Let E be a completely regular topological space and U an open subset of E. We will consider classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of maps.

Definition 2.1. We say $F \in D(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{D}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E and $\overline{U}$ denotes the closure of U in E.

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Definition 2.2. We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, E)$ of F.

We fix a $\Phi \in B(\overline{U}, E)$.

Definition 2.3. We say $F \in A_{\partial U}(\overline{U}, E)$ (respectively $F \in D_{\partial U}(\overline{U}, E)$) if $F \in A(\overline{U}, E)$ (respectively $F \in D(\overline{U}, E)$) with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.4. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$ if for any selection $\Psi \in D(\overline{U}, E)$ of F and any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

Remark 2.1. Note if $F \in A_{\partial U}(\overline{U}, E)$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$ and if $\Psi \in D(\overline{U}, E)$ is any selection of F then there exists an $x \in U$ with $\Psi(x) \cap \Phi(x) \neq \emptyset$ (take $J = \Psi$ in Definition 2.4; note for $x \in \partial U$ that $\Psi(x) \cap \Phi(x) \subseteq F(x) \cap \Phi(x) = \emptyset$). Also note if $\Psi(x) \cap \Phi(x) \neq \emptyset$ for $x \in U$ then $\emptyset \neq \Psi(x) \cap \Phi(x) \subseteq F(x) \cap \Phi(x)$.

We begin with a new nonlinear alternative of Leray–Schauder type

Theorem 2.1. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $F \in A(\overline{U}, E)$ and let $G \in A_{\partial U}(\overline{U}, E)$ be Φ–essential in $A_{\partial U}(\overline{U}, E)$. For any selection $\Lambda \in D(\overline{U}, E)$ (respectively $\Psi \in D(\overline{U}, E)$) of G (respectively F) assume there exists a map $H^{\Lambda,\Psi} : \overline{U} \times [0,1] \to 2^E$ with $H^{\Lambda,\Psi}(\,.\,, \eta(\,.\,)) \in D(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t^{\Lambda,\Psi}(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\left\{x \in \overline{U} :\ \Phi(x) \cap H^{\Lambda,\Psi}(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$ is compact (respectively closed) and $H_1^{\Lambda,\Psi} = \Psi$, $H_0^{\Lambda,\Psi} = \Lambda$; here $H_t^{\Lambda,\Psi}(x) = H^{\Lambda,\Psi}(x,t)$. Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$ (in fact $\Phi(x) \cap \Psi(x) \neq \emptyset$).*

Proof: Suppose $\Phi(x) \cap F(x) = \emptyset$ for $x \in \partial U$ (otherwise we are finished). Let $\Lambda \in D(\overline{U}, E)$ (respectively $\Psi \in D(\overline{U}, E)$) be any selection of G (respectively F). Choose the map $H^{\Lambda,\Psi}$ as in the statement of Theorem 2.1. Let

$$\Omega = \left\{x \in \overline{U} :\ \Phi(x) \cap H^{\Lambda,\Psi}(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Note $\Omega \neq \emptyset$ since $H_0^{\Lambda,\Psi} = \Lambda$ and G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ (see Remark 2.1). Also Ω is compact (respectively closed) if E is a completely regular (respectively normal) topological space. Next note $\Omega \cap \partial U = \emptyset$ (note $H_0^{\Lambda,\Psi} = \Lambda$ and $\Phi(x) \cap G(x) = \emptyset$ for $x \in \partial U$ since $G \in A_{\partial U}(\overline{U}, E)$ and note $H_1^{\Lambda,\Psi} = \Psi$ and we assumed $\Phi(x) \cap F(x) \neq \emptyset$ for $x \in \partial U$). Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map J by $J(x) = H^{\Lambda,\Psi}(x, \mu(x))$. Now $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Lambda|_{\partial U}$ (note if $x \in \partial U$ then $J(x) = H_0^{\Lambda,\Psi}(x) = \Lambda(x)$ and $J(x) \cap \Phi(x) = \Lambda(x) \cap \Phi(x) = \emptyset$). Now since G

is Φ–essential in $A_{\partial U}(\overline{U},E)$ then there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H^{\Lambda,\Psi}_{\mu(x)}(x) \cap \Phi(x) \neq \emptyset$). Thus $x \in \Omega$ so $\mu(x) = 1$. As a result $H^{\Lambda,\Psi}_1(x) \cap \Phi(x) \neq \emptyset$ i.e. $\Psi(x) \cap \Phi(x) \neq \emptyset$ i.e. $F(x) \cap \Phi(x) \neq \emptyset$ since Ψ is a selection of F. $\square$

Remark 2.2. We say $F \in MA(\overline{U},E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U},E)$, and we say $F \in MA_{\partial U}(\overline{U},E)$ if $F \in MA(\overline{U},E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$. Now let $F \in MA_{\partial U}(\overline{U},E)$. We say $F : \overline{U} \to 2^E$ is Φ–essential in $MA_{\partial U}(\overline{U},E)$ if for every map $J \in MA_{\partial U}(\overline{U},E)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. The argument in Theorem 2.1 immediately yields the following result:

Let E be a completely regular (respectively normal) topological space, U an open subset of E, $F \in MA(\overline{U},E)$ and let $G \in MA_{\partial U}(\overline{U},E)$ be Φ–essential in $MA_{\partial U}(\overline{U},E)$. Suppose there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,,\eta(\,.\,)) \in MA(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\{x \in \overline{U} :\ \Phi(x) \cap H(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$ is compact (resp. closed) and $H_1 = F$, $H_0 = G$; here $H_t(x) = H(x,t)$. Then there exists $x \in \overline{U}$ with $\Phi(x) \cap F(x) \neq \emptyset$.

All the results in this paper (with the exception of Theorem 2.2) have corresponding results for the class MA (we leave the obvious statements to the reader).

Now we consider a special case of Theorem 2.1 using the standard homotopy between two maps. Note topological vector spaces are automatically completely regular.

Corollary 2.1. *Let E be a topological vector space, U an open subset of E, $F \in A(\overline{U},E)$ and let $G \in A_{\partial U}(\overline{U},E)$ be Φ–essential in $A_{\partial U}(\overline{U},E)$. For any selection $\Lambda \in D(\overline{U},E)$ (respectively $\Psi \in D(\overline{U},E)$) of G (respectively F) suppose*

$$\text{(2.1)} \qquad \begin{cases} \mu(\,.\,)\,\Psi(\,.\,) + (1-\mu(\,.\,))\,\Lambda(\,.\,) \in D(\overline{U},E) \ \ \textit{for any} \\ \textit{continuous map } \ \mu : \overline{U} \to [0,1] \ \ \textit{with} \ \ \mu(\partial U) = 0 \end{cases}$$

and

$$\text{(2.2)} \qquad \begin{cases} K = \{x \in \overline{U} :\ \Phi(x) \cap [t\,\Psi(x) + (1-t)\,\Lambda(x)] \neq \emptyset \ \ \textit{for} \\ \textit{some} \ \ t \in [0,1]\} \ \textit{is compact} \end{cases}$$

hold. Then either

(A1). there exists $x \in \overline{U}$ with $F(x) \cap \Phi(x) \neq \emptyset$,

or

(A2). there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $\Phi(x) \cap [\lambda\,F(x) + (1-\lambda)\,G(x)] \neq \emptyset$,

hold.

Proof: Let $\Lambda \in D(\overline{U}, E)$ (respectively $\Psi \in D(\overline{U}, E)$) be any selection of G (respectively F) and let

$$H^{\Lambda,\Psi}(x,t) = t\,\Psi(x) + (1-t)\,\Lambda(x).$$

Suppose $(A2)$ does not hold. Now if $x \in \partial U$ and $\lambda \in (0,1)$ then $\Phi(x) \cap [\lambda\, F(x) + (1-\lambda)\, G(x)] = \emptyset$ so $\Phi(x) \cap [\lambda\,\Psi(x) + (1-\lambda)\,\Lambda(x)] = \emptyset$ since $\lambda\,\Psi(x) + (1-\lambda)\,\Lambda(x) \subseteq \lambda\, F(x) + (1-\lambda)\, G(x)$. Then Theorem 2.1 guarantees that there exists a $x \in \overline{U}$ with $F(x) \cap \Phi(x) \neq \emptyset$ (i.e. $(A1)$ holds). $\square$

Remark 2.3. If E in Corollary 2.1 is normal then (2.2) can be replaced by: K is closed.

Remark 2.3. In Corollary 2.1 we could have $(A1)$ and $A2)$ as:

(A1). there exists $x \in \overline{U}$ with $\Psi(x) \cap \Phi(x) \neq \emptyset$,

(A2). there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $\Phi(x) \cap [\lambda\,\Psi(x) + (1-\lambda)\,\Lambda(x)] \neq \emptyset$.

Before we discuss Corollary 2.1 we first recall the DKT maps from the literature. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in DKT(Z, W)$ if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibres $S^{-1}(w) = \{z : w \in S(z)\}$ are open (in Z) for each $w \in W$.

Recall $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$, $F \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, E)$ of F. However in some situations we have a map $F : \overline{U} \to 2^E$, $F \in \mathbf{A}(\overline{U}, E)$ but we do not know if there exists a selection $\Psi \in D(\overline{U}, E)$ of F. For example let $F : \overline{U} \to 2^E$ with $F \in DKT(\overline{U}, E)$ a compact map (here $\mathbf{A}(\overline{U}, E)$ denotes the class of compact DKT maps from $\overline{U}$ to 2^E). Suppose $D(\overline{U}, E)$ denotes the class of single valued continuous compact maps. If $\overline{U}$ is paracompact then we know [4] that there exists a selection $\Psi \in D(\overline{U}, E)$ of F. However if $\overline{U}$ is not necessarily paracompact is it possible to obtain a Leray–Schauder alternative of Corollary 2.1 type? Our next theorem is a result of this type and is motivated in part by [9].

Theorem 2.2. *Let E be a topological vector space, U an open subset of E, $F : \overline{U} \to 2^E$ and $G : \overline{U} \to 2^E$. Suppose there exists a set $K \subseteq E$ with $F(\overline{U}) \subseteq K$, $G(\overline{U}) \subseteq K$, $F \in A(\overline{U \cap L(K)} \cap L(K), L(K))$ and let*

$$G \in A_{\partial_{L(K)}\,(U \cap L(K))}(\overline{U \cap L(K)} \cap L(K), L(K))$$

be Φ–essential in $A_{\partial_{L(K)}\,(U \cap L(K))}(\overline{U \cap L(K)} \cap L(K), L(K))$; here $L(K)$ is the linear span of K (i.e. the smallest linear subspace of E that contains K) and

$\partial_{L(K)}(U \cap L(K))$ denotes the boundary of $U \cap L(K)$ in $L(K)$. For any selection $\Lambda \in D(\overline{U \cap L(K)} \cap L(K), L(K))$ (respectively $\Psi \in D(\overline{U \cap L(K)} \cap L(K), L(K))$) of G (respectively F) suppose

$$(2.3) \qquad \begin{cases} \mu(\,.\,)\,\Psi(\,.\,) + (1-\mu(\,.\,))\,\Lambda(\,.\,) \in D(\overline{U \cap L(K)} \cap L(K), L(K)) \\ \text{for any continuous map } \mu : \overline{U} \to [0,1] \text{ with } \mu(\partial U) = 0 \end{cases}$$

$$(2.4) \qquad \begin{cases} \Omega = \{x \in \overline{U \cap L(K)} \cap L(K) : \ \Phi(x) \cap [t\,\Psi(x) + (1-t)\,\Lambda(x)] \neq \emptyset \\ \text{for some } t \in [0,1]\} \text{ is compact} \end{cases}$$

and

$$(2.5) \qquad \begin{cases} \text{for } x \in \partial_{L(K)}(U \cap L(K)) \text{ and } \lambda \in (0,1) \text{ we have} \\ \Phi(x) \cap [\lambda\,F(x) + (1-\lambda)\,G(x)] = \emptyset. \end{cases}$$

Then there exists $x \in \overline{U}$ with $F(x) \cap \Phi(x) \neq \emptyset$.

Proof: Note $U \cap L(K)$ is an open subset of $L(K)$ and $\overline{U \cap L(K)}^{\,L(K)} = \overline{U \cap L(K)} \cap L(K)$; here $\overline{U \cap L(K)}^{\,L(K)}$ denotes the closure of $U \cap L(K)$ in $L(K)$. From Corollary 2.1 we see that there exists a $x \in \overline{U \cap L(K)} \cap L(K)$ ($\subseteq \overline{U}$) with $F(x) \cap \Phi(x) \neq \emptyset$. $\square$

Remark 2.5. If $L(K)$ is normal then we can replace (2.4) with: Ω is closed. For example if $L(K)$ is paracompact then $L(K)$ is normal (recall paracompact spaces are normal).

Remark 2.6. Note in Theorem 2.2 we could replace (2.5) with either

$$\begin{cases} \text{for } x \in \partial_{L(K)}(U \cap L(K)) \text{ and } \lambda \in (0,1) \text{ we have} \\ \Phi(x) \cap [\lambda\,\Psi(x) + (1-\lambda)\,\Lambda(x)] = \emptyset \end{cases}$$

or

$$\begin{cases} \text{for } x \in \partial U \cap L(K) \text{ and } \lambda \in (0,1) \text{ we have} \\ \Phi(x) \cap [\lambda\,\Psi(x) + (1-\lambda)\,\Lambda(x)] = \emptyset \end{cases}$$

or

$$\begin{cases} \text{for } x \in \partial U \text{ and } \lambda \in (0,1) \text{ we have} \\ \Phi(x) \cap [\lambda\,F(x) + (1-\lambda)\,G(x)] = \emptyset. \end{cases}$$

This is immediate since $\partial_{L(K)}(U \cap L(K)) \subseteq \partial U$; to see this note

$$\begin{aligned} \partial_{L(K)}(U \cap L(K)) &= (\overline{U \cap L(K)} \cap L(K)) \setminus (U \cap L(K)) \\ &\subseteq (\overline{U} \cap L(K)) \setminus (U \cap L(K)) \\ &= (\overline{U} \cap L(K)) \setminus U \,\cup\, (\overline{U} \cap L(K)) \setminus L(K) \\ &= (\overline{U} \cap L(K)) \setminus U \subseteq \overline{U} \setminus U = \partial U. \end{aligned}$$

Now let us return to our example before Theorem 2.2. Let $F : \overline{U} \to 2^E$ with $F \in DKT(\overline{U}, E)$ a compact map. Now let K be a compact set with $F(\overline{U}) \subseteq K$ and note $L(K)$ is paracompact (see for example [5]). Now if we show $F \in DKT(\overline{U \cap L(K)} \cap L(K), L(K))$ then [4] guarantees that (recall closed subsets of paracompact spaces are paracompact) there exists a selection $\Psi \in D(\overline{U \cap L(K)} \cap L(K), L(K))$ of F. Since $F \in DKT(\overline{U}, E)$ then there exists a map $\theta : \overline{U} \to E$ with $co\,(\theta(x)) \subseteq F(x)$ for $x \in \overline{U}$, $\theta(x) \neq \emptyset$ for each $x \in \overline{U}$ and $\theta^{-1}(y) = \{z \in \overline{U} : y \in \theta(z)\}$ is open (in $\overline{U}$) for each $y \in E$. Let $\theta^\star$ denote the restriction of θ to $\overline{U \cap L(K)} \cap L(K)$. Note $co\,(\theta^\star(x)) \subseteq F(x)$ for $x \in \overline{U \cap L(K)} \cap L(K)$ and $\theta^\star(x) \neq \emptyset$ for each $x \in \overline{U \cap L(K)} \cap L(K)$. If $y \in L(K)$ then (note $\overline{U \cap L(K)} \cap L(K) \cap \overline{U} = \overline{U \cap L(K)} \cap L(K)$ since $\overline{U \cap L(K)} \subseteq \overline{U}$),

$$\begin{aligned}
(\theta^\star)^{-1}(y) &= \{z \in \overline{U \cap L(K)} \cap L(K) :\, y \in \theta^\star(z)\} \\
&= \{z \in \overline{U \cap L(K)} \cap L(K) :\, y \in \theta(z)\} \\
&= \overline{U \cap L(K)} \cap L(K) \cap \{z \in \overline{U} :\, y \in \theta(z)\} \\
&= \overline{U \cap L(K)} \cap L(K) \cap \theta^{-1}(y)
\end{aligned}$$

which is open in $\overline{U \cap L(K)} \cap L(K)$. Thus $F \in DKT(\overline{U \cap L(K)} \cap L(K), L(K))$.

Using Corollary 2.1 we present some Birkhoff–Kellogg type theorems (our results improve those in [2, 3, 10]).

Theorem 2.3. *Let E be a topological vector space, U an open subset of E, $H : \overline{U} \to 2^E$, and $G \in A_{\partial U}(\overline{U}, E)$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$. Assume*

$$\text{there exists } \mu \in \mathbf{R} \text{ with } \mu\, H(x) \cap G(x) = \emptyset \text{ for } x \in \overline{U}. \tag{2.6}$$

Let $F = \mu\, H$ and suppose $F \in A(\overline{U}, E)$. In addition for any selection $\Lambda \in D(\overline{U}, E)$ (respectively $\Psi \in D(\overline{U}, E)$) of G (respectively F) suppose (2.1) *and* (2.2) *hold. Then there exists $x \in \partial U$ and $\lambda \in (0, 1)$ with $\Phi(x) \cap [\lambda\, \mu\, H(x) + (1 - \lambda)\, G(x)] \neq \emptyset$.*

PROOF: The result follows from Corollary 2.1 since (2.6) guarantees that for $x \in \overline{U}$ we have $F(x) \cap G(x) = \emptyset$. $\square$

Theorem 2.4. *Let E be a topological vector space, U an open subset of E, $\Theta : \partial U \to 2^E$, and $G \in A_{\partial U}(\overline{U}, E)$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$. Assume*

$$\left\{ \begin{array}{l} \partial U \text{ is a retract of } \overline{U} \text{ i.e. there exists a} \\ \text{retraction } r_0 : \overline{U} \to \partial U \end{array} \right. \tag{2.7}$$

and

$$\text{there exists } \mu \in \mathbf{R} \text{ with } \mu\, \Theta(\partial U) \cap G(\overline{U}) = \emptyset. \tag{2.8}$$

Let $F = \mu\,\Theta\,r_0$ and suppose $F \in A(\overline{U}, E)$. In addition for any selection $\Lambda \in D(\overline{U}, E)$ (respectively $\Psi \in D(\overline{U}, E)$) of G (respectively F) assume (2.1) *and* (2.2) *hold. Then there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $\Phi(x) \cap [\lambda\,\mu\,\Theta(x) + (1-\lambda)\,G(x)] \neq \emptyset$.*

PROOF: Note $H = \Theta\,r_0 : \overline{U} \to 2^E$ and note (2.8) guarantees that $\mu\,H(\overline{U}) \cap G(\overline{U}) = \emptyset$. Now Theorem 2.3 guarantees that there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $\Phi(x) \cap [\lambda\,\mu\,\Theta\,r_0(x) + (1-\lambda)\,G(x)] \neq \emptyset$ i.e. $\Phi(x) \cap [\lambda\,\mu\,\Theta(x) + (1-\lambda)\,G(x)] \neq \emptyset$. $\square$

The ideas above could be applied to other natural situations. Let E be a Hausdorff topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\,L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\,L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear single valued map with $L + T : dom\,L \to Y$ a bijection; for convenience we say $T \in H_L(E, Y)$.

Definition 2.5. We say $F \in D(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}\,(F+T) \in D(\overline{U}, E)$ (respectively $(L+T)^{-1}\,(F+T) \in B(\overline{U}, E)$).

Definition 2.6. We say $F \in A(\overline{U}, Y; L, T)$ if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}\,(F+T) \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, Y; L, T)$ of F.

We <u>fix</u> a $\Phi \in B(\overline{U}, Y; L, T)$.

Definition 2.7. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ (respectively we say $F \in D_{\partial U}(\overline{U}, Y; L, T)$) if $F \in A(\overline{U}, Y; L, T)$) (respectively $F \in D(\overline{U}, Y; L, T)$)) with $(L+T)^{-1}\,(F+T)(x) \cap (L+T)^{-1}\,(\Phi+T)(x) = \emptyset$ for $x \in \partial U$.

Definition 2.8. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F : \overline{U} \to 2^Y$ is L–Φ–essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for any selection $\Psi \in D(\overline{U}, Y; L, T)$ of F and any map $J \in D_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = \Psi|_{\partial U}$ there exists $x \in U$ with $(L+T)^{-1}\,(J+T)\,(x) \cap (L+T)^{-1}\,(\Phi+T)\,(x) \neq \emptyset$.

We now present a Leray–Schauder alternative in this setting.

Theorem 2.5. *Let E be a topological vector space, Y a topological vector space, U an open subset of E, $L : dom\,L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, $F \in A(\overline{U}, Y; L, T)$ and let $G \in A_{\partial U}(\overline{U}, Y; L, T)$ be L–Φ–essential in $A_{\partial U}(\overline{U}, Y; L, T)$. For any selection $\Lambda \in D(\overline{U}, Y; L, T)$ (respectively $\Psi \in D(\overline{U}, Y; L, T)$) of G (respectively F) suppose there exists a map $H^{\Lambda,\Psi}$ defined on $\overline{U} \times [0,1]$ with values in Y with $(L+T)^{-1}\,(H^{\Lambda,\Psi}(\,.\,, \eta(\,.\,)) + T(\,.\,)) \in D(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}\,(H^{\Lambda,\Psi}_t + T)(x) \cap (L+T)^{-1}\,(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and*

$t \in (0,1)$, $H_1^{\Lambda,\Psi} = \Psi$, $H_0^{\Lambda,\Psi} = \Lambda$ and

$$\begin{aligned}\Omega &= \{x \in \overline{U} : (L+T)^{-1}(\Phi+T)(x) \cap (L+T)^{-1}(H_t^{\Lambda,\Psi}+T)(x) \neq \emptyset \\ &\quad \text{for some } t \in [0,1]\}\end{aligned}$$

is compact; here $H_t^{\Lambda,\Psi}(x) = H^{\Lambda,\Psi}(x,t)$. Then there exists $x \in \overline{U}$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$.

PROOF: Suppose $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for $x \in \partial U$ (otherwise we are finished). Let $\Lambda \in D(\overline{U},Y;L,T)$ (respectively $\Psi \in D(\overline{U},Y;L,T)$) be any selection of G (respectively F). Choose the map $H^{\Lambda,\Psi}$ and the set Ω as in the statement of Theorem 2.5. Note $\Omega \neq \emptyset$ (since $H_0^{\Lambda,\Psi} = \Lambda$ and $G \in A_{\partial U}(\overline{U},Y;L,T)$ be L–Φ–essential in $A_{\partial U}(\overline{U},Y;L,T)$), Ω is compact and $\Omega \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map J by $J(x) = H^{\Lambda,\Psi}(x,\mu(x)) = H_{\mu(x)}^{\Lambda,\Psi}(x)$. Now $J \in D_{\partial U}(\overline{U},Y;L,T)$ and $J|_{\partial U} = \Lambda|_{\partial U}$ (if $x \in \partial U$ then $J(x) = H_0^{\Lambda,\Psi}(x) = \Lambda(x)$). Now since G is L–Φ–essential in $A_{\partial U}(\overline{U},Y;L,T)$ there exists $x \in U$ with $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$. Thus $x \in \Omega$ so $\mu(x) = 1$. As a result $(L+T)^{-1}(H_1^{\Lambda,\Psi}+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$ i.e. $(L+T)^{-1}(\Psi+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$. The result follows since Ψ is a selection of F. $\square$

Remark 2.7. If in Theorem 2.5 the space E is additionally normal then the assumption that Ω is compact can be replaced by: Ω is closed.

From Theorem 2.5 it is easy to obtain analogues of Corollary 2.1 and Theorem's 2.3 and 2.4 (we leave this to the reader)

Next we present a Furi–Pera type result (see [1, 6, 8] and the references therein).

Theorem 2.6. *Let E be a metrizable topological vector space, Q a closed subset of E, $\Phi \in A(Q,E)$ and $F \in A(Q,E)$. Assume the following conditions hold:*

$$(2.9) \qquad \begin{cases} \text{there exists a retraction } r : E \to Q \text{ with} \\ r(z) \in \partial Q \text{ for } z \in E \setminus Q \end{cases}$$

and

$$(2.10) \qquad \begin{cases} \text{for any selection } \Psi \in D(Q,E) \text{ of } F \text{ assume } \Psi r \in D(E,E), \\ \Psi r \text{ has a fixed point in } E \text{ and } \Omega = \{x \in E : x \in \Psi r(x)\} \\ \text{is compact.} \end{cases}$$

For $i \in \{1,2,...\}$ let $U_i = \{x \in E : d(x,Q) < \frac{1}{i}\}$; here d is the metric associated with E. Suppose for each $i \in \{1,2,...\}$ and for any selection $\Psi \in$

$D(Q,E)$ of F and any selection $\phi \in D(Q,E)$ of Φ we have the following:

$$\Psi\, r \in D(\overline{U_i}, E) \quad and \quad \phi\, r \in D(\overline{U_i}, E) \tag{2.11}$$

$$\begin{cases} either\ (A1).\ there\ exists\ \ x \in \overline{U_i}\ \ with\ \ x \in \Psi\, r(x), \\ or\ (A2).\ there\ exists\ \ x \in \partial U_i\ \ and\ \ \lambda \in (0,1) \\ with\ \ x \in \lambda\, \Psi\, r(x) + (1-\lambda)\, \phi\, r(x),\ \ hold \end{cases} \tag{2.12}$$

$$\begin{cases} \{x \in E:\ x \in \lambda\, \Psi\, r(x) + (1-\lambda)\, \phi\, r(x)\ \ for \\ some\ \ \lambda \in [0,1]\}\ \ is\ compact \end{cases} \tag{2.13}$$

$$\begin{cases} if\ \ \{(x_j, \lambda_j\}_{j=1}^{\infty}\ \ is\ a\ sequence\ in\ \ \partial U_i \times [0,1] \\ converging\ to\ \ (x,\lambda)\ \ with\ \ x \in \partial Q\ \ and \\ x_j \in \lambda_j\, \Psi\, r\,(x_j) + (1-\lambda_j)\, \phi\, r\,(x_j),\ \ then \\ x \in \lambda\, \Psi\, r\,(x) + (1-\lambda)\, \phi\, r\,(x) = \lambda\, \Psi\,(x) + (1-\lambda)\, \phi\,(x) \end{cases} \tag{2.14}$$

and

$$\begin{cases} if\ \ \{(x_j, \lambda_j\}_{j=1}^{\infty}\ \ is\ a\ sequence\ in\ \ \partial Q \times [0,1]\ \ converging \\ to\ \ (x,\lambda)\ \ with\ \ x \in \lambda\, \Psi\,(x) + (1-\lambda)\, \phi\,(x)\ \ and\ \ 0 \le \lambda < 1, \\ then\ \ \{\lambda_j\, \Psi\,(x_j) + (1-\lambda_j)\, \phi\,(x_j)\} \subseteq Q\ \ for\ \ j\ \ sufficiently\ large. \end{cases} \tag{2.15}$$

Then F has a fixed point in Q.

PROOF: Let $\Psi \in D(Q,E)$ be a selection of F and let Ω be as in (2.10). Now $\Omega \neq \emptyset$ is compact. We claim $\Omega \cap Q \neq \emptyset$. To do this we argue by contradiction. Suppose that $\Omega \cap Q = \emptyset$. Then since Ω is compact and Q is closed there exists $\delta > 0$ with $dist(\Omega, Q) > \delta$. Choose $m \in \{1,2,...\}$ with $1 < \delta\, m$ and let (as in the statement of the theorem) $U_i = \{x \in E:\ d(x,Q) < \frac{1}{i}\}$ for $i \in \{m, m+1, ...\}$.

Fix $i \in \{m, m+1, ...\}$. Since $dist(\Omega, Q) > \delta$ we see that $\Omega \cap \overline{U_i} = \emptyset$. Let $\phi \in D(Q,E)$ be a selection of Φ. Now (2.12) guarantees that there exists $\lambda_i \in (0,1)$ and $y_i \in \partial U_i$ with $y_i \in \lambda_i\, \Psi\, r(y_i) + (1-\lambda_i)\, \phi\, r(y_i)$. Since $y_i \in \partial U_i$ we have

$$\{\lambda_i\, \Psi\, r(y_i) + (1-\lambda_i)\, \phi\, r(y_i)\} \not\subseteq Q \quad \text{for} \quad i \in \{m, m+1, ...\}. \tag{2.16}$$

Let

$$K = \{x \in E:\ x \in \lambda\, \Psi\, r(x) + (1-\lambda)\, \phi\, r(x)\ \ \text{for some}\ \ \lambda \in [0,1]\}.$$

Now $K \neq \emptyset$ is compact (see (2.10) and (2.13)) and this together with

$$d(y_j, Q) = \frac{1}{j} \quad \text{and} \quad |\lambda_j| \le 1 \quad \text{for} \quad j \in \{m, m+1, ...\}$$

implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \to y^\star \in \partial Q$. Now (2.14) implies $y^\star \in \lambda^\star \Psi\, r(y^\star) + (1-\lambda^\star)\,\phi\, r(y^\star)$ i.e. $y^\star \in \lambda^\star \Psi\,(y^\star) + (1-\lambda^\star)\,\phi\,(y^\star)$ since $r(y^\star) = y^\star$. If $\lambda^\star = 1$ then $y^\star \in \Psi\, r(y^\star) = \Psi\,(y^\star)$ which contradicts $\Omega \cap Q = \emptyset$. Thus $0 \leq \lambda^\star < 1$. Now (2.15) with $x_j = r(y_j)$ (note $y_j \in \partial U_j$ so $r(y_j) \in \partial Q$) and $x = y^\star = r(y^\star)$ implies

$$\{\lambda_j\, \Psi\, r(y_j) + (1-\lambda_j)\,\phi\, r(y_j)\} \subseteq Q \quad \text{for} \quad j \quad \text{sufficiently large.}$$

This contradicts (2.16). Thus $\Omega \cap Q \neq \emptyset$ so there exists $x \in Q$ with $x \in \Psi\, r(x) = \Psi(x) \subseteq F(x)$. $\square$

Remark 2.8. Suppose in Theorem 2.6 we change (2.9) to: there exists a retraction $r : E \to Q$. Then the result in Theorem 2.6 again holds provided (2.15) is changed to

$$\begin{cases} \text{if} \ \ \{(x_j, \lambda_j\}_{j=1}^{\infty} \ \ \text{is a sequence in} \ \ Q \times [0,1] \ \ \text{converging} \\ \text{to} \ \ (x, \lambda) \ \ \text{with} \ \ x \in \lambda\, \Psi\,(x) + (1-\lambda)\,\phi\,(x) \ \ \text{and} \ \ 0 \leq \lambda < 1, \\ \text{then} \ \ \{\lambda_j\, \Psi\,(x_j) + (1-\lambda_j)\,\phi\,(x_j)\} \subseteq Q \ \ \text{for} \ \ j \ \ \text{sufficiently large.} \end{cases}$$

Remark 2.9. Technically we do not need to assume (2.11) in Theorem 2.6.

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